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Berry's phase and Hannay's angle from quantum canonical transformations

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Abstract. Using an 'action-angle' coherent states formalism, introduced by the authors in a preceding paper, it is shown that Berry's phase and Hannay's angle can both be derived from the same quantum unitary transformation; their relationship is easily established in this framework.

1. Introduction

In this paper we report on a result which directly issues from the action-angle coherent states formalism (AACSF) introduced in a preceding paper [1] devoted to the problem of the relationship between the quantum Berry's phase [2] and the classical Hannay's angle [3]. This relationship was first studied by Berry using semi-classical approximations (Maslov method) [4]. The main interest of the AACSF is that it clarifies the similarity between this quantum phase and classical angle, because the latter may find a quantum interpretation in terms of coherent states. In particular, in exactly the same way as for Berry's phase, the AACSF allows Hannay's angle to be deduced from the minimization of a distance in the (quantum) Hilbert space, this minimization leading to a geometrical transport equation for classical tori [1].

It is interesting to complete the study of the similarity between Berry's phase and Hannay's angle by considering their relationship from the point of view of quantum canonical transformations. This approach is quite natural in the classical case, where the connection of Hannay's angle with the generating functional of a time-dependent canonical transformation to action-angle variables is well known [4]. In the quantum context, unitary transformations have also been evoked but essentially to discuss their influence on the decomposition of the total phase into a dynamical part and a geometrical one (Berry's phase) [5, 6].

Using the AACSF, we show in this paper that Berry's phase and Hannay's angle can both be derived from the same quantum unitary transformation. In the classical limit this transformation becomes the expected canonical transformation to action-angle variables. Berry's phase and Hannay's angle are obtained by taking the mean values of the operator relationship between the initial and transformed Hamiltonians respectively, in the energy eigenstates and in the 'action-angle' coherent states.

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2. Action-angle coherent states

Let $H_S(\boldsymbol{X})$ be the parameter-dependent Hamiltonian of a (one degree of freedom) quantum system, in the Schrödinger representation. Its spectrum $\{E_n(\boldsymbol{X})\}$ is supposed to be discrete ($n = 0, 1, \ldots, \infty$) and non-degenerate in the range of variation of the parameters $\boldsymbol{X}$ considered in the following. For each value of $\boldsymbol{X}$ we make a definite choice (continuous with respect to $\boldsymbol{X}$) of the eigenstates $\{|n, \boldsymbol{X}\rangle\}$.

In [1] the coherent states associated with the Hamiltonian $H_S(\boldsymbol{X})$ are defined by :

$$|\alpha, \boldsymbol{X}\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n, \boldsymbol{X}\rangle. \tag{1}$$

Their main interest is that, in the classical limit ($\hbar \to 0, |\alpha| \to \infty, |\alpha|^2\hbar$ finite), the complex number α reads $\hbar^{-1/2} I^{1/2} \exp(-i\theta)$ in terms of the action I (characterizing a trajectory of the Hamiltonian $H_S(\boldsymbol{X})$ in phase space) and of the angle θ (specifying a point on that trajectory). These states generalize the usual coherent states $|\alpha\rangle$ associated with the Hamiltonian of the harmonic oscillator. It is important for the following, to notice that a change of Hamiltonian induces a change of coordinates in the classical phase space: a given point can be associated with the ray $\overline{|\alpha_1, \boldsymbol{X}_1\rangle}$ as well as the ray $\overline{|\alpha_2, \boldsymbol{X}_2\rangle}$, according to the choice of Hamiltonians $H_S(\boldsymbol{X}_1)$ or $H_S(\boldsymbol{X}_2)$. Coherent states corresponding to the same point in phase space will be said to be classically equivalent: $|\alpha_1, \boldsymbol{X}_1\rangle \sim |\alpha_2, \boldsymbol{X}_2\rangle$. The canonical transformations which we shall consider next are special cases of such changes of action-angle coordinates $\alpha_1(I_1, \theta_1) \to \alpha_2(I_2, \theta_2)$.

Another property that the states $|\alpha, \boldsymbol{X}\rangle$ share with the states $|\alpha\rangle$ concerns the relationship between quantum and classical observables [7]. In particular, the mean value $\langle \alpha, \boldsymbol{X}|O|\alpha, \boldsymbol{X}\rangle$ and the quantity $\langle \alpha, \boldsymbol{X}|[O_1, O_2]|\alpha, \boldsymbol{X}\rangle$, relative to quantum observables $O, O_1, O_2, \ldots$, respectively become the classical observable $o(I, \theta, \boldsymbol{X})$ and the Poisson bracket $\{o_1, o_2\}_{\mathrm{PB}}$ in the classical limit. The only difference from the usual situation is that the classical observables now appear naturally as functions of the action-angle variables associated with the Hamiltonian $H_S(\boldsymbol{X})$ instead of the coordinates p and q. Mean values of a quantum observable in classically equivalent coherent states, are equal but lead to expressions of the corresponding classical observable, in terms of different action-angle coordinates.

3. Quantum canonical transformation

We now consider the situation where the parameters $\boldsymbol{X}$ vary slowly with time (adiabatic hypothesis). The quantum adiabatic theorem says that the evolved state $U(t)|n, \boldsymbol{X}(0)\rangle$ of an initial eigenstate $|n, \boldsymbol{X}(0)\rangle$ of the Hamiltonian at time zero is an eigenstate $\exp(i\varphi_n(t))|n, \boldsymbol{X}(t)\rangle$ of the Hamiltonian at time t. Let $V(t)$ be the unitary operator so that:

$$V(t)|n, \boldsymbol{X}(0)\rangle = |n, \boldsymbol{X}(t)\rangle. \tag{2}$$

Then, as already noted [6,8], adiabaticity implies that the operator $V^+(t)U(t) = U'(t)$ is diagonal in the basis $\{|n, \boldsymbol{X}(0)\rangle\}$ with eigenvalues $\exp i\varphi_n(t)$.

The interest in a canonical transformation is the simplification of the equations of motion; we define it so that the Schrödinger evolution operator $U(t)$ is replaced by $U'(t)$. In the Heisenberg picture, where the states are considered as fixed, this transformation results in a change from the 'old' Heisenberg observables $O_{\mathrm{H}}(t) = U^{+}(t)O_{\mathrm{S}}U(t)$ to the transformed ones:

$$O'_{\mathrm{H}}(t) = U'^{+}(t)O_{\mathrm{S}}U'(t) \qquad (U'(t) = V^{+}(t)U(t)). \tag{3}$$

Let us remark that the change $O_{\mathrm{H}}(t) \to O'_{\mathrm{H}}(t)$ preserves the commutation relations between observables.

In the classical limit, this unitary transformation corresponds to a change of coordinates for the representative point M_t in the phase space of the state of the system at time t: its action-angle coordinates $\alpha_0(t)$ associated with the Hamiltonian $H_{\mathrm{S}}(\boldsymbol{X}(0))$ are transformed into those $\alpha(t)$ associated with the Hamiltonian $H_{\mathrm{S}}(\boldsymbol{X}(t))$. This can be shown in the following way. Let $U(t)|\alpha_0(0), \boldsymbol{X}(0)\rangle$ be the evolved state of an initial coherent state $|\alpha_0(0), \boldsymbol{X}(0)\rangle$; this state is classically equivalent to the states $|\alpha_0(t), \boldsymbol{X}(0)\rangle$ and $|\alpha(t), \boldsymbol{X}(t)\rangle$ associated respectively with $H_{\mathrm{S}}(\boldsymbol{X}(0))$ and $H_{\mathrm{S}}(\boldsymbol{X}(t))$ which also represents the point M_t. Then, introducing the 'annihilation operator' A_{S} at time zero, defined by $A_{\mathrm{S}}|\alpha_0, \boldsymbol{X}(0)\rangle = \alpha_0|\alpha_0, \boldsymbol{X}(0)\rangle$ for any α_0, one finds that

$$\langle\alpha_0(0), \boldsymbol{X}(0)|A_{\mathrm{H}}(t)|\alpha_0(0), \boldsymbol{X}(0)\rangle = \langle\alpha_0(t), \boldsymbol{X}(0)|A_{\mathrm{S}}|\alpha_0(t), \boldsymbol{X}(0)\rangle = \alpha_0(t)$$

while

$$\langle\alpha_0(0), \boldsymbol{X}(0)|A'_{\mathrm{H}}(t)|\alpha_0(0), \boldsymbol{X}(0)\rangle = \langle\alpha(t), \boldsymbol{X}(t)|V^{+}(t)A_{\mathrm{S}}V(t)|\alpha(t), \boldsymbol{X}(t)\rangle = \alpha(t).$$

In the classical limit, all observables are functions of A_{S} (and A_{S}^{+}) only and thus the quantum unitary transformation $O_{\mathrm{H}}(t) \to O'_{\mathrm{H}}(t)$ amounts to the time-dependent classical canonical transformation from the action-angle variables $\alpha_0(t)$ to $\alpha(t)$.

4. Berry's phase and Hannay's angle

As is well known, a canonical transformation leads to a transformed Hamiltonian $K(t)$ which governs the time evolution of the new observables: $\dot{O}'_{\mathrm{H}}(t) = \mathrm{i}\hbar^{-1}[K(t), O'_{\mathrm{H}}(t)]$. A short calculation shows that $K(t) = \mathrm{i}\hbar U'^{+}(t)\dot{U}'(t)$ may also be written:

$$K(t) = U^{+}(t)H_{\mathrm{S}}(\boldsymbol{X}(t))U(t) - \mathrm{i}\hbar U^{+}(t)\dot{V}(t)V^{+}(t)U(t). \tag{4}$$

In this decomposition of $K(t)$ the second term which is only present when the transformation (i.e. V) is time-dependent, is at the origin of Berry's phase and Hannay's angle.

Let us first show that one can obtain the Berry's phase by taking the mean value of both sides of the relationship (4) into the states $|n, \boldsymbol{X}(0)\rangle$. From (2) and the property that $U'(t)$ is diagonal in the basis $\{|n, \boldsymbol{X}(0)\rangle\}$ one immediately gets:

$$\dot{\varphi}_n(t) = -\hbar^{-1}E_n(\boldsymbol{X}(t)) + \mathrm{i}\left\langle n, \boldsymbol{X}(t)\left|\frac{\partial}{\partial t}\right|n, \boldsymbol{X}(t)\right\rangle. \tag{5}$$

When integrated in time, this relationship shows the decomposition of the total phase φ_n into a dynamical part and an extra one, Berry's phase γ_n^{B}.

In order to derive the classical counterpart of (4) it is natural now to take its mean value into a coherent state $|\alpha_0(0), \boldsymbol{X}(0)\rangle$. From (2) and the classical equivalence $U(t)|\alpha_0(0), \boldsymbol{X}(0)\rangle \sim |\alpha(t), \boldsymbol{X}(t)\rangle$ one obtains:

$$k(I, \theta, \boldsymbol{X}(t)) = h(I, \theta, \boldsymbol{X}(t)) - \mathrm{i}\hbar\langle\alpha(t), \boldsymbol{X}(t)|\nabla_{\boldsymbol{X}}|\alpha(t), \boldsymbol{X}(t)\rangle\dot{\boldsymbol{X}}. \qquad (6)$$

Taking into account the remarks of sections 2 and 3 on the classical limit of observables, one realizes that $h(I, \theta, \boldsymbol{X}(t))$ and $k(I, \theta, \boldsymbol{X}(t))$ are respectively the original and transformed classical Hamiltonians expressed in terms of the action-angle variables at time t. Since we have proved that the transformation (3) describes a change of action-angle coordinates from $\alpha_0(t)$ to $\alpha(t)$, the second term in the right-hand side of (6) must be the partial time derivative of the generating functional of this transformation. (This is shown in detail in the appendix.) The angular velocity $\dot{\theta}$ on the trajectories is obtained through the relationship $\dot{\theta} = (\partial/\partial I)k(I, \theta, \boldsymbol{X}(t))$. Thus it also appears as the sum of a dynamical part $(\partial/\partial I)h(I, \theta, \boldsymbol{X}(t))$ and an extra one whose time integration leads to the Hannay's angle θ_I^{H} [4].

Finally, let us show that, in the classical limit, the relationship between Berry's phase and Hannay's angle can be easily established in the present framework. The starting point rests on the remark that the average $(2\pi)^{-1}\int_0^{2\pi}\langle\alpha, \boldsymbol{X}|O|\alpha, \boldsymbol{X}\rangle \mathrm{d}\theta$ of a coherent state expectation value of an observable O over the argument θ of α, i.e. over a trajectory is, in this limit, equal to the expectation value $\langle n, \boldsymbol{X}|O|n, \boldsymbol{X}\rangle$ into the energy eigenstate such that $|\alpha|^2 = n$ [1]. Then, up to the factor $\hbar^{-1}$, (5) is nothing but the average of (6) over a trajectory. Due to the adiabatic hypothesis, Hannay's velocity $\dot{\theta}_I^{\mathrm{H}}(t)$ is slowly varying and can also be obtained from the averaged version of (6) over a trajectory. Therefore one recovers the standard relation $\theta_I^{\mathrm{H}} = -\hbar(\partial/\partial I)\gamma_n^{\mathrm{B}}$.

Appendix

The usual expression [4] of the second term in the right-hand side of (6) can be explicitly derived if, taking into account classical equivalence, we write the state $|\alpha, \boldsymbol{X}\rangle$ under the form $\exp(\mathrm{i}\hbar^{-1}\Psi(\alpha_0(\alpha, \boldsymbol{X}), \boldsymbol{X}))|\alpha_0(\alpha, \boldsymbol{X}), \boldsymbol{X}(0)\rangle$. Then the term under consideration reads:

$$(\nabla_{\boldsymbol{X}}\Psi - \mathrm{i}\hbar\langle\alpha_0(\alpha, \boldsymbol{X}), \boldsymbol{X}(0)|\nabla_{\boldsymbol{X}}|\alpha_0(\alpha, \boldsymbol{X}), \boldsymbol{X}(0)\rangle)\dot{\boldsymbol{X}}.$$

Since the states $|\alpha_0, \boldsymbol{X}(0)\rangle$ are related to the coherent states $|\alpha_0\rangle$ of the harmonic oscillator by a unitary transformation (the one which brings the states $|n, \boldsymbol{X}(0)\rangle$ onto the states $|n\rangle$) one can use the relation [7]: $\langle\alpha_0|\alpha_0 + d\alpha_0\rangle = 1 + \mathrm{i}\hbar^{-1}q\,\mathrm{d}p$ to obtain the final form: $(\nabla_{\boldsymbol{X}}S - p\nabla_{\boldsymbol{X}}q)\dot{\boldsymbol{X}}$. In this last expression the coordinates q, p and $S = \Psi - qp$ are functions of the 'running' coordinates $\alpha(I, \theta)$ and of the parameters $\boldsymbol{X}$ and the partial derivatives with respect to $\boldsymbol{X}$ are taken for fixed I and θ.

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